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Analytic solutions of the temperature distribution in Blasius viscous flow problems

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We apply a new analytic technique, namely the homotopy analysis method, to give an analytic approximation of temperature distributions for a laminar viscous flow over a semi-infinite plate. An explicit analytic solution of the temperature distributions is obtained in general cases and recurrence formulae of the corresponding constant coefficients are given. In the cases of constant plate temperature distribution and constant plate heat flux, the first-order derivative of the temperature on the plate at the 30th order of approximation is given. The convergence regions of these two formulae are greatly enlarged by the Padé technique. They agree well with numerical results in a very large region of Prandtl number $1 \leqslant Pr \leqslant 50$ and therefore can be applied without interpolations.

1. Introduction

Most problems in fluid mechanics are nonlinear. Thus, it is important to develop efficient methods to solve them. Since the electronic computer appeared, numerical techniques for nonlinear partial differential equations (PDEs) have been developing quickly. However, up to now, it is still difficult to obtain analytic approximations of nonlinear PDEs, even though there exist high-performance supercomputers and much high-quality symbolic computation software such as MATHEMATICA, MATHLAB, MAPLE and so on. The reason might be that we do not have a satisfactory analytic tool valid for highly nonlinear PDEs. Perturbation techniques (Kevorkian & Cole 1991; Nayfeh 1973, 1979; Rand & Armbruster 1987) are widely applied to nonlinear problems. However, it is well known that perturbation techniques are essentially based on small perturbation quantities, and the so-called 'small parameter assumption' of perturbation techniques greatly restricts their applications.

Recently, Liao (1997, 1999*a*–*c*, 2002) developed a new kind of analytic method for nonlinear problems, called the homotopy analysis method. Different from perturbation techniques, the homotopy analysis method is valid even for nonlinear problems whose governing equations and/or boundary conditions do not contain any small parameters at all. The homotopy analysis method also provides us with great freedom to select proper base functions to approximate solutions of nonlinear problems. Thus, it can be applied to more nonlinear problems in science and engineering. For example, Liao applied the homotopy analysis method to give explicit, purely analytical solutions of Blasius viscous flow (Liao 1999*a*) and Falkner–Skan viscous flow (Liao 1999*b*). Recently, Liao (2002) successfully applied the homotopy analysis method to a well-known classical problem in fluid mechanics, i.e. the viscous flow past a sphere, and gave, the first time, a tenth-order drag coefficient formula. This illustrates the great potential of the homotopy analysis method. A so-called generalized Taylor series can also be derived in the framework of the homotopy analysis method, which logically contains the classical Taylor series and provides a simple way to enlarge the convergence region of perturbation approximations (Liao 1999*c*). This also provides us with a mathematical, logical guarantee of the validity of the homotopy analysis method. This encourages us to employ the homotopy analysis method to more complicated problems, while further improving it.

In our previous works mentioned above, the homotopy analysis method was applied to solve equations with only one unknown function. In this paper, we apply it to a nonlinear problem containing two unknown functions. As a starting point, we still consider the two-dimensional laminar viscous flow past a flat plate (Rand & Armbruster 1987), governed by

$$f'''(\eta) + \tfrac{1}{2} f(\eta) f''(\eta) = 0 \tag{1.1}$$

with boundary conditions

$$f(0) = f'(0) = 0, \quad f'(+\infty) = 1, \tag{1.2}$$

where $f(\eta)$ is related to the stream function ψ by $f(\eta) = \psi/\sqrt{\nu U x}$, η is given by $\eta = y\sqrt{U/(\nu x)}$, U is the constant velocity at infinity, ν is the kinematic viscosity coefficient, and x and y are the two independent coordinates. To consider the related heat transfer problem, we define a non-dimensional temperature

$$\theta = \frac{T_w - T}{T_w - T_\infty}, \tag{1.3}$$

where T denotes the dimensional temperature, and T_w and T_∞ are temperatures at the boundary and infinity, respectively. If the wall-temperature distribution satisfies the power law

$$T_w(x) - T_\infty = C x^\kappa, \tag{1.4}$$

where C is a constant and κ is a real number, then, in the absence of frictional heat, $\theta(\eta)$ satisfies a second-order differential equation

$$\theta''(\eta) + \frac{Pr}{2} f(\eta)\theta'(\eta) + \kappa\, Pr\, f'(\eta)[1 - \theta(\eta)] = 0 \tag{1.5}$$

under the boundary conditions

$$\theta(0) = 0, \quad \theta(+\infty) = 1, \tag{1.6}$$

where the prime denotes the derivative with respect to η, $Pr = k/(\mu C_p)$ the Prandtl number, C_p the specific heat at constant pressure, k the thermal conductivity and μ the viscosity. Note that $\kappa = 0$ corresponds to a flat plate with a constant temperature, and $\kappa = 1/2$ a constant heat flux.

We emphasize that (1.5) and (1.6) have no solutions when $Pr = 0$. Thus, one cannot use the Prandtl number Pr as the perturbation quantity to obtain perturbation approximations. However, as shown in Liao (1999*a*), if η is used as a perturbation quantity, the perturbation solution of (1.1) and (1.2) is valid in a rather restricted region $|\eta| < 5.69$. Thus, the convergence radius of the perturbation solution of (1.5) and (1.6) is certainly not greater than 5.69. Thus, an outer solution had to be given and the so-called matching perturbation technique had to be applied. As a result, the corresponding perturbation approximations are not uniformly valid in the whole region of the flow. Besides, such a perturbation solution contains an unknown quantity $\theta'(0)$, which is however a function of the Prandtl number Pr and had to be given by numerical methods. Thus, this kind of perturbation approximation is semi-analytic and semi-numerical. To the best of our knowledge, there does not exist any purely analytic formula for the temperature distribution of the Blasius viscous flow nor especially an analytic formula for $\theta'(0)$.

In the next section, we apply the homotopy analysis method to give an explicit analytic solution of the above-mentioned problem in general cases. In § 3, we give analytic formula for $\theta'(0)$ at the 30th-order of approximation for both $\kappa = 0$ and $\kappa = 1/2$, and apply the Padé approximation technique to greatly enlarge their convergence regions. In § 4, some discussion and conclusions are given.

2. Mathematical formulations

To apply the homotopy analysis method to the problem considered, we first select two kinds of auxiliary linear operators

$$\mathrm{L}_f = \frac{\partial^3}{\partial\eta^3} + \beta\frac{\partial^2}{\partial\eta^2}, \quad \beta > 0, \tag{2.1}$$

$$\mathrm{L}_\theta = \frac{\partial^2}{\partial\eta^2} + \beta\frac{\partial}{\partial\eta}, \quad \beta > 0. \tag{2.2}$$

Then, we construct a family of PDEs

$$(1-p)\mathrm{L}_f[F(\eta,p) - f_0(\eta)] = \hbar p\left[\frac{\partial^3 F(\eta,p)}{\partial\eta^3} + \tfrac{1}{2}F(\eta,p)\frac{\partial^2 F(\eta,p)}{\partial\eta^2}\right], \tag{2.3}$$

$$(1-p)\mathrm{L}_\theta[\Theta(\eta,p) - \theta_0(\eta)] = \hbar_\theta p\left\{\frac{\partial^2\Theta(\eta,p)}{\partial\eta^2} + \frac{Pr}{2}F(\eta,p)\frac{\partial\Theta(\eta,p)}{\partial\eta}\right.$$

$$\left. + \kappa Pr[1-\Theta(\eta,p)]\frac{\partial F(\eta,p)}{\partial\eta}\right\}, \tag{2.4}$$

with boundary conditions

$$F(0,p) = F_\eta(0,p) = 0, \quad F_\eta(+\infty,p) = 1, \tag{2.5}$$

$$\Theta(0,p) = 0, \quad \Theta(+\infty,p) = 1, \tag{2.6}$$

where F_η denotes the first-order derivative of $F(\eta,p)$ with respect to η, $p \in [0,1]$ is an imbedding parameter, $\hbar \neq 0$ and $\hbar_\theta \neq 0$ are two non-zero auxiliary parameters, and $f_0(\eta)$ and $\theta_0(\eta)$ are two initial guess approximations of $f(\eta)$ and $\theta(\eta)$, respectively. Considering the boundary conditions (1.2) and (1.6), we select

$$f_0(\eta) = \eta - \frac{1-\exp(-\beta\eta)}{\beta} \tag{2.7}$$

and

$$\theta_0(\eta) = 1 - \exp(-\beta\eta). \tag{2.8}$$

When $p=0$, we have the solution

$$F(\eta,0)=f_0(\eta) \tag{2.9}$$

and

$$\Theta(\eta,0)=\theta_0(\eta). \tag{2.10}$$

When $p=1$, equations (2.3)–(2.6) are the same as (1.1), (1.5), (1.2) and (1.6), respectively, so that

$$F(\eta,1)=f(\eta) \tag{2.11}$$

and

$$\Theta(\eta,1)=\theta(\eta). \tag{2.12}$$

The initial guess approximations $f_0(\eta)$ and $\theta_0(\eta)$, the two auxiliary linear operators L_f and L_θ, and the two auxiliary parameters $\hbar$ and $\hbar_\theta$ are assumed to be selected such that equations (2.3)–(2.6) have solutions at each point $p \in [0,1]$, and also $F(\eta,p)$, $\Theta(\eta,p)$ can be expressed in Maclaurin series

$$F(\eta,p)=F(\eta,0)+\sum_{k=1}^{+\infty}\frac{p^k}{k!}\left.\frac{\partial^k F(\eta,p)}{\partial p^k}\right|_{p=0}, \tag{2.13}$$

$$\Theta(\eta,p)=\Theta(\eta,0)+\sum_{k=1}^{+\infty}\frac{p^k}{k!}\left.\frac{\partial^k \Theta(\eta,p)}{\partial p^k}\right|_{p=0}. \tag{2.14}$$

Defining

$$\varphi_0(\eta)=F(\eta,0)=f_0(\eta), \quad \varphi_k(\eta)=\frac{1}{k!}\left.\frac{\partial^k F(\eta,p)}{\partial p^k}\right|_{p=0} \quad (k>0) \tag{2.15}$$

and

$$\psi_0(\eta)=\Theta(\eta,0)=\theta_0(\eta), \quad \psi_k(\eta)=\frac{1}{k!}\left.\frac{\partial^k \Theta(\eta,p)}{\partial p^k}\right|_{p=0} \quad (k>0), \tag{2.16}$$

we have due to (2.9)–(2.14) that

$$F(\eta,p)=\varphi_0(\eta)+\sum_{k=1}^{+\infty}\varphi_k(\eta)p^k, \tag{2.17}$$

$$\Theta(\eta,p)=\psi_0(\eta)+\sum_{k=1}^{+\infty}\psi_k(\eta)p^k. \tag{2.18}$$

Obviously, the convergence region of the above two series depends upon the auxiliary linear operators L_f and L_θ, and the two non-zero auxiliary parameters $\hbar$ and $\hbar_\theta$. If all of them are selected so that (2.17) and (2.18) converge at $p=1$, we have due to (2.11) and (2.12) that

$$f(\eta)=\varphi_0(\eta)+\sum_{m=1}^{+\infty}\varphi_m(\eta), \tag{2.19}$$

$$\theta(\eta)=\psi_0(\eta)+\sum_{m=1}^{+\infty}\psi_m(\eta). \tag{2.20}$$

Equations (2.19) and (2.20) provide us with a relationship between the initial guess solutions $f_0(\eta), \theta_0(\eta)$ and the unknown solutions $f(\eta), \theta(\eta)$, respectively.

In order to give the governing equations of $\varphi_m(\eta)$ and $\psi_m(\eta)$ $(m \geqslant 1)$, we first differentiate m times the two sides of equations (2.3)–(2.6) about the embedding parameter p, then set $p = 0$, and finally divide them by $m!$. In this way, we obtain the governing equations for $\varphi_m(\eta)$ and $\psi_m(\eta)$ $(m \geqslant 1)$:

$$\mathrm{L}_f[\varphi_m - \chi_m \varphi_{m-1}] = \hbar R_m(\eta), \tag{2.21}$$

$$\mathrm{L}_\theta[\psi_m - \chi_m \psi_{m-1}] = \hbar_\theta W_m(\eta), \tag{2.22}$$

with boundary conditions

$$\varphi_m(0) = \varphi_m'(0) = \varphi_m'(+\infty) = 0, \tag{2.23}$$

$$\psi_m(0) = \psi_m(+\infty) = 0, \tag{2.24}$$

where

$$R_m(\eta) = \varphi_{m-1}'''(\eta) + \frac{1}{2}\sum_{k=0}^{m-1} \varphi_k''(\eta)\varphi_{m-1-k}(\eta), \tag{2.25}$$

$$\begin{aligned} W_m(\eta) = {} & \psi_{m-1}''(\eta) + \kappa\, Pr\, \varphi_{m-1}'(\eta) \\ & + Pr \sum_{k=0}^{m-1} \left[\tfrac{1}{2}\varphi_{m-1-k}(\eta)\psi_k'(\eta) - \kappa \varphi_{m-1-k}'(\eta)\psi_k(\eta)\right], \end{aligned} \tag{2.26}$$

and

$$\chi_m = \begin{cases} 0, & \text{when } m = 1, \\ 1, & \text{otherwise.} \end{cases} \tag{2.27}$$

We emphasize that (2.21)–(2.24) are linear for all $m \geqslant 1$. Also, the left-hand sides of (2.21) and (2.22) are governed by the same linear operator L_f and L_θ, respectively, for all $m \geqslant 1$. These linear equations can be easily solved, especially by means of symbolic calculation software such as MATHEMATICA, MATHLAB, MAPLE and so on. Liao (1999*a*) solved equations (2.21), (2.23) and gave the following explicit analytic solution:

$$\varphi_m = \sum_{k=0}^{m+1} \exp(-k\beta\eta) \sum_{i=0}^{2(m+1-k)} \lambda_{m,k}^i b_{m,k}^i \eta^i, \tag{2.28}$$

where

$$\lambda_{m,k}^i = \begin{cases} 0, & \text{when } m = k = 0, \quad i \geqslant 2, \\ 0, & \text{when } m > 0, \quad k = 0, \quad i \geqslant 1, \\ 0, & \text{when } k > m+1, \\ 0, & \text{when } i > 2(m+1-k), \\ 1, & \text{otherwise,} \end{cases} \tag{2.29}$$

and the constant coefficients $b_{m,k}^i$ are calculated by the recurrence formulas (for details, refer to Liao 1999*a*). Thus, in this paper, we regard $b_{m,k}^i$ as known coefficients. Similarly, we apply the symbolic computation software MATHEMATICA to solve the first several dozen governing equations (2.22) under the boundary condition (2.24) successively in order $(m = 1, 2, 3, \ldots)$. After analysing these approximations, we find that $\psi_m(\eta)$ can be explicitly expressed in the general form

$$\psi_m = \sum_{k=0}^{m+1} \exp(-k\beta\eta) \sum_{i=0}^{2(m+1-k)} \Lambda_{m,k}^i \eta^i \sum_{j=0}^{m} S_{m,k}^{i,j} (Pr)^j, \tag{2.30}$$

where

$$\Lambda^i_{m,k} = \begin{cases} 0, & \text{when } k = 0, \quad i^2 + m^2 > 0, \\ 0, & \text{when } k > m + 1, \\ 0, & \text{when } i > 2(m + 1 - k), \\ 0, & \text{when } m < 0, \\ 1, & \text{otherwise,} \end{cases} \tag{2.31}$$

and $S^{i,j}_{m,k}$ are constant coefficients, which can be calculated by the following recurrence formulas:

$$S^{0,m}_{m,1} = -\hbar_\theta \Omega^{0,m}_{m,n} \mu^0_{n,0}, \tag{2.32}$$

$$S^{0,j}_{m,1} = \chi_m \Lambda^0_{m-1,1} S^{0,j}_{m-1,1} - \hbar_\theta \sum_{n=2}^{m+1} \sum_{q=0}^{2(m-n+1)} \Omega^{q,j}_{m,n} \mu^q_{n,0}, \quad \text{when } j \neq m, \tag{2.33}$$

$$S^{k,m}_{m,1} = \hbar_\theta \sum_{q=k-1}^{2m-1} \Omega^{q,m}_{m,1} \mu^q_{1,k}, \quad \text{when } 1 \leqslant k \leqslant 2m, \tag{2.34}$$

$$S^{k,j}_{m,1} = \chi_m \Lambda^k_{m-1,1} S^{k,j}_{m-1,1} + \hbar_\theta \sum_{q=k-1}^{2m-1} \Omega^{q,j}_{m,1} \mu^q_{1,k}, \quad \text{when } 1 \leqslant k \leqslant 2m, \quad j \neq m, \tag{2.35}$$

$$S^{k,m}_{m,n} = \hbar_\theta \sum_{q=k}^{2(m-n+1)} \Omega^{q,m}_{m,n} \mu^q_{n,k}, \quad \text{when } 2 \leqslant n \leqslant m + 1, \tag{2.36}$$

$$S^{k,j}_{m,n} = \chi_m \Lambda^k_{m-1,n} S^{k,j}_{m-1,n} + \hbar_\theta \sum_{q=k}^{2(m-n+1)} \Omega^{q,j}_{m,n} \mu^q_{n,k}, \quad \text{when } 2 \leqslant n \leqslant m + 1, \quad j \neq m, \tag{2.37}$$

$$\mu^q_{1,k} = -\frac{q!}{k!\,\beta^{q-k+2}}, \quad 0 \leqslant k \leqslant q + 1, \quad q \geqslant 0, \tag{2.38}$$

$$\mu^q_{n,k} = \left(\frac{q!}{k!}\right) \frac{(n-1)}{[\beta(n-1)]^{q-k+2}} \left[1 - \left(1 - \frac{1}{n}\right)^{q-k+1}\right],$$
$$0 \leqslant k \leqslant q, \quad q \geqslant 0, \quad n \geqslant 2, \tag{2.39}$$

$$\Omega^{q,0}_{m,n} = \Lambda^q_{m-1,n} C^{q,0}_{m-1,n}, \tag{2.40}$$

$$\Omega^{q,1}_{m,n} = \Lambda^q_{m-1,n}[\chi_m C^{q,1}_{m-1,n} + \kappa(a^q_{m-1,n} - S^{q,0}_{m-1,n})] + \Delta^{q,0}_{m,n}, \tag{2.41}$$

$$\Omega^{q,m}_{m,n} = \Delta^{q,m-1}_{m,n} - \kappa \Lambda^q_{m-1,n} S^{q,m-1}_{m-1,n}, \tag{2.42}$$

$$\Omega^{q,j}_{m,n} = \Lambda^q_{m-1,n}[C^{q,j}_{m-1,n} - \kappa S^{q,j-1}_{m-1,n}] + \Delta^{q,j-1}_{m,n}, \quad \text{when } 1 < j < m, \tag{2.43}$$

$$\Delta^{q,l}_{m,n} = \frac{1}{2} \sum_{k=l}^{m-1} \sum_{j=J_0}^{J_1} \sum_{i=I_0}^{I_1} \lambda^{q-i}_{m-1-k,n-j} b^{q-i}_{m-1-k,n-j} A^{i,l}_{k,j}$$
$$- \sum_{k=0}^{m-1-l} \sum_{j=J_0}^{J_1} \sum_{i=I_0}^{I_1} \kappa \Lambda^{q-i}_{m-1-k,n-j} S^{q-i,l}_{m-1-k,n-j} a^i_{k,j}, \tag{2.44}$$

$$a^i_{m,k} = (i+1)\lambda^{i+1}_{m,k} b^{i+1}_{m,k} - (k\beta)\lambda^i_{m,k} b^i_{m,k}, \tag{2.45}$$

$$A_{m,k}^{i,j} = (i+1)A_{m,k}^{i+1}S_{m,k}^{i+1,j} - (k\beta)A_{m,k}^{i}S_{m,k}^{i,j}, \tag{2.46}$$

$$C_{m,k}^{i,j} = (i+2)(i+1)A_{m,k}^{i+2}S_{m,k}^{i+2,j} - 2(k\beta)(i+1)A_{m,k}^{i+1}S_{m,k}^{i+1,j} + (k\beta)^2 A_{m,k}^{i}S_{m,k}^{i,j}, \tag{2.47}$$

with the definitions of I_0, I_1, J_0 and J_1 as follows:

$$J_0 = \max\{1, n+k-m\}, \quad J_1 = \min\{n, k+1\}, \tag{2.48}$$

$$I_0 = \max\{0, q-2(m-k-n+j)\}, \quad I_1 = \min\{q, 2(k-j+1)\}. \tag{2.49}$$

Due to the initial guess (2.8), the first three coefficients are

$$S_{0,0}^{0,0} = 1, \quad S_{0,1}^{0,0} = -1, \quad S_{0,0}^{1,0} = 0. \tag{2.50}$$

Using these three coefficients and the above recurrence formulae, we can calculate all coefficients $S_{m,n}^{k,j}$ successively.

It should be emphasized that we give here the analytic solution in the form of recurrence formulae, just like the definitions (in series) of some fundamental functions such as $\sin(x), \cos(x)$ and so on. Liao (1999*a*) has also strictly proved that a series given by the homotopy analysis method must be one of its exact/correct solutions, as long as it is convergent. So, theoretically speaking, if the values of $\hbar$ and $\hbar_\theta$ are properly selected so that the series (2.20) is convergent, one can gain as accurate result as possible, just like when calculating the value of some fundamental functions like $\sin(x), \cos(x)$ and so on.

Finally, we point out again that when $Pr = 0$ the original equations (1.5) and (1.6) have no solution at all. This fact suggests two conclusions. First, it is impossible to use Pr as a small parameter to obtain a perturbation expression for the temperature distribution of Blasius flow. Second, the solution given by the homotopy analysis method should have no physical meaning at $Pr = 0$ and/or might be convergent very slowly near $Pr = 0$. This is indeed true, as shown in the next section.

3. Analytic formula for $\theta'(0)$

From the viewpoint of engineering, it is important to know the value of $\theta'(0)$. By (2.20) and (2.30), we have its Mth-order approximation

$$\theta'(0) = \sum_{m=0}^{M} \Psi'_m(0) = \sum_{j=0}^{M} \sigma_{M,j}(Pr)^j, \tag{3.1}$$

where

$$\sigma_{m,j} = \sum_{n=j}^{m}\sum_{k=0}^{n+1}\left(A_{n,k}^{1}S_{n,k}^{1,j} - \kappa\beta A_{n,k}^{0}S_{n,k}^{0,j}\right). \tag{3.2}$$

Writing

$$s_m = \sum_{j=0}^{m} \sigma_{m,j}(Pr)^j, \tag{3.3}$$

the convergence region of the series

$$s_0, s_1, s_2, s_3, \ldots \tag{3.4}$$

depends upon the values of the auxiliary parameters $\beta, \hbar$ and $\hbar_\theta$. Liao (1999*a*) pointed out that (2.28) is convergent in the region $\beta \geqslant 4$ and $-1 \leqslant \hbar < 0$. For simplicity, we select here $\beta = 5, \hbar = -1$ to ensure that the series (2.28) is convergent. Then, the convergence of (3.4) is dependent only upon the value of $\hbar_\theta$. Obviously, $\sigma_{m,j}$ is a function of κ. We found that, for any value of κ, s_m contains the term $\sigma_{m,0} = \beta(1+\hbar_\theta)^m$, say,

$$s_m = \beta(1+\hbar_\theta)^m + \sum_{j=1}^{m} \sigma_{m,j}(Pr)^j. \tag{3.5}$$

Thus, $|1+\hbar_\theta| < 1$ is a necessary condition to ensure that the series

$$\lim_{m\to+\infty} \sum_{j=0}^{m} \sigma_{m,j}(Pr)^j \tag{3.6}$$

is convergent. Our calculations indicate that the above series indeed converges when $-1 \leqslant \hbar_\theta < 0$. We are not surprised at this result. Notice that the series (2.28) for the stream function converges when $-1 \leqslant \hbar < 0$. Here, we emphasize once again that it has been proved in general cases that a series given by the homotopy analysis method must be one of its exact/correct solutions as long as it is convergent (Liao 1999*a*). So, we only need to select proper values of $\hbar$ and $\hbar_\theta$ to ensure the convergence of (3.6), and calculate as many terms in (3.6) as we can, so as to give as accurate results of $\theta'(0)$ as possible.

Let q denote the heat flux. Then, due to the definition of $\theta(\eta)$ and η, we have

$$q(x) = k\frac{\partial T}{\partial y}\bigg|_{y=0} = \frac{kU_\infty}{\sqrt{\nu x}}[\mathrm{T}_\infty - T_w(x)]\theta'(0). \tag{3.7}$$

Substituting (1.4) into (3.7) gives

$$q(x) = \frac{kCU_\infty\theta'(0)}{\sqrt{\nu}} x^{\kappa-1/2}. \tag{3.8}$$

So, the temperature distribution (1.4) is equivalent to the heat flux distribution (3.8). In this paper, we consider two special cases, $\kappa = 0$ and $\kappa = 1/2$, corresponding to a constant temperature distribution ($\kappa = 0$)

$$T_\infty - T_w(x) = C, \tag{3.9}$$

and a constant heat flux distribution ($\kappa = 1/2$)

$$q(x) = \frac{kCU_\infty\theta'(0)}{\sqrt{\nu}}, \tag{3.10}$$

respectively.

(a) $\kappa = 0$

Similar to all of the previous results given by Liao (1997, 1999*a*–*c*), the convergence region of the series (3.4) depends upon the value of $\hbar_\theta$, where $-1 \leqslant \hbar_\theta < 0$. Our calculations indicate that the closer $\hbar_\theta$ is to zero the larger the convergence region of the series (3.4). For example, when $\hbar_\theta = -3/4$, the 30th-order approximation of $\theta'(0)$ is convergent in the region near $Pr \leqslant 5$. When $\hbar_\theta = -1/2$, it converges in the region near $Pr \leqslant 6$. This is consistent with our previous results given by the homotopy analysis method for other problems, i.e. the related convergence regions tend to infinity as $\hbar$ ($\hbar < 0$) tends to zero. We have mathematically proved this in some special cases, although we have not given a rigorous mathematical proof in general. However, the closer the value of $\hbar_\theta$ is to zero, the more terms we need to obtain an accurate enough result. This is mainly because the constant term $\beta(1+\hbar_\theta)^m$ of the series (3.6) decays rather slowly when $\hbar_\theta$ is near zero.

The 30th-order approximations provide us with sufficient information about $\theta'(0)$. Thus, we can apply some well-known techniques to enlarge its convergence region. In this paper, the Padé approximation is applied. Notice that, when $-1 \leqslant \hbar_\theta < 0$, $\beta(1+\hbar_\theta)^m \to 0$ as $m \to \infty$. Thus, the term $\beta(1+\hbar_\theta)^m$ can be seen as a kind of error and should be deleted. We emphasize once again that the original equations (1.5) and (1.6) have no solutions when $Pr = 0$. Due to (3.6), $\theta'(0) = \beta(1+\hbar_\theta)^m$ decays to zero when $Pr = 0$. This however has no physical meaning at all. Thus, it is improper to use $Pr = 0$ as the centre of expansion of the corresponding Padé approximation. In this paper, we apply the symbolic software MATHEMATICA (version 3.0) to obtain $\theta'(0)$ at up to the 30th-order of approximation, then delete the decaying term $\sigma_{m,0} = \beta(1+\hbar_\theta)^m$, and finally rewrite the remainder in the following (15,15) Padé approximation:

$$\theta'(0) = \frac{\sum_{k=0}^{15} a_k (Pr-1)^k}{1 + \sum_{k=1}^{15} a_{15+k}(Pr-1)^k}, \tag{3.11}$$

where $a_k (k = 0, 1, 2, 3, \ldots, 30)$ are constant coefficients, whose values are dependent upon the selection of $\hbar_\theta$. The comparisons of the numerical results with the corresponding analytic solutions given by (3.11) for $\hbar_\theta = -0.75$ and -0.5 are shown in table 1 and figure 1. In the region $1 \leqslant Pr \leqslant 25$, the Padé approximations for both $\hbar_\theta = -0.5$ and -0.75 give nearly the same results, which agree quite well with the numerical ones. However, in the region $Pr < 1$, none of the Padé approximations can give good enough agreement with the numerical results. This is mainly because there does not exist a solution of (1.5) and (1.6) when $Pr = 0$, so that $Pr = 0$ is a singular point. In order to obtain accurate results near $Pr = 0$, more terms are needed. In the region $Pr > 25$, the closer $\hbar_\theta$ is to zero, the better the Padé approximations. This is mainly because the convergence region of the series (3.6) becomes larger when $\hbar_\theta$ is closer to zero. Notice that even at $Pr = 50$, the Padé approximation when $\hbar_\theta = -0.5$ has less than 0.5% relative error compared to the numerical result. Thus, the Padé expression (3.11) for $\hbar_\theta = -0.5$ can be used as an accurate analytic formula for $\theta'(0)$ in the region $1.0 \leqslant Pr \leqslant 50$. For this purpose, we list the corresponding coefficients $a_k (k = 0, 1, 2, \ldots, 30)$ as follows:

$$\begin{array}{ll}
a_0 = 3.2698447817 \times 10^{-1} & a_{16} = 1.5121905700 \\
a_1 = 6.2603294361 \times 10^{-1} & a_{17} = 4.5910120619 \times 10^{-1} \\
a_2 = 2.8478117096 \times 10^{-1} & a_{18} = -8.9895054049 \times 10^{-1} \\
a_3 = -2.8281960009 \times 10^{-1} & a_{19} = -1.3299608157 \\
a_4 = -5.4496831628 \times 10^{-1} & a_{20} = -9.7417491657 \times 10^{-1} \\
a_5 = -4.4365427880 \times 10^{-1} & a_{21} = -4.7988127826 \times 10^{-1} \\
a_6 = -2.3753774956 \times 10^{-1} & a_{22} = -1.7324386574 \times 10^{-1} \\
a_7 = -9.2265979634 \times 10^{-2} & a_{23} = -4.7420046998 \times 10^{-2} \\
a_8 = -2.7162933942 \times 10^{-3} & a_{24} = -9.9529836680 \times 10^{-3} \\
a_9 = -6.1351110859 \times 10^{-3} & a_{25} = -1.5959788956 \times 10^{-3} \\
a_{10} = -1.0659763898 \times 10^{-3} & a_{26} = -1.9190978794 \times 10^{-4} \\
a_{11} = -1.3996725333 \times 10^{-4} & a_{27} = -1.6617708138 \times 10^{-5} \\
a_{12} = -1.3522404214 \times 10^{-5} & a_{28} = -9.5203521752 \times 10^{-7} \\
a_{13} = -8.9095185876 \times 10^{-7} & a_{29} = -2.9312817722 \times 10^{-8} \\
a_{14} = -3.5230778079 \times 10^{-8} & a_{30} = -1.9612748551 \times 10^{-10} \\
a_{15} = -4.7033736379 \times 10^{-10} &
\end{array}$$

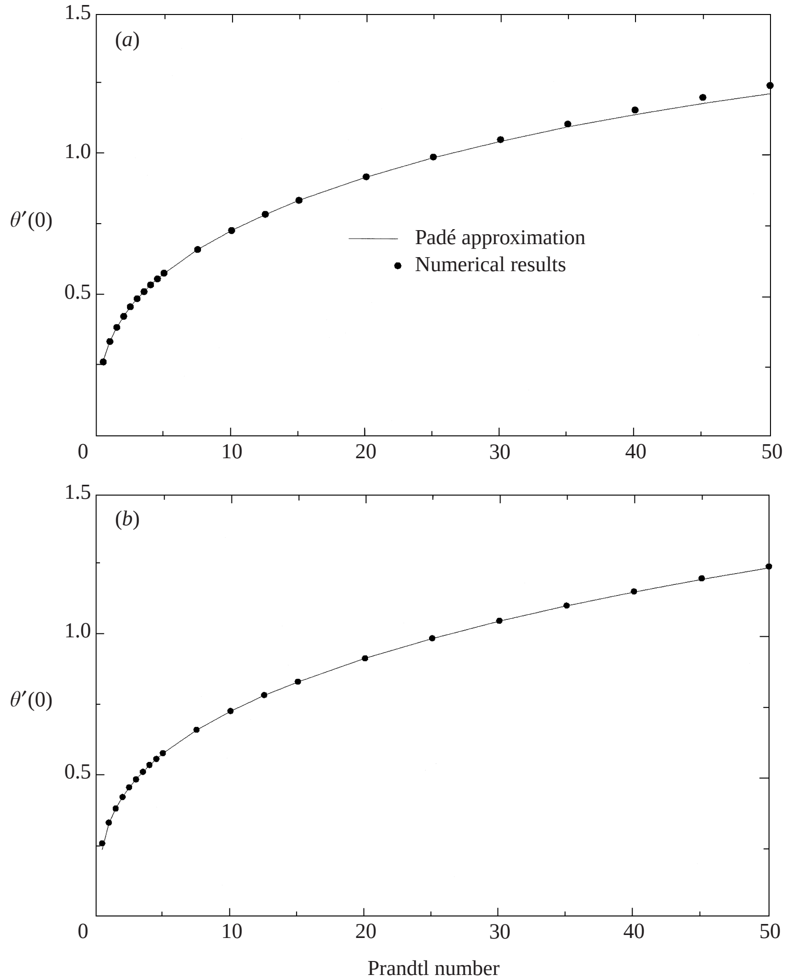

FIGURE 1. Comparisons of the Padé approximation (3.11) for $\theta'(0)$ with numerical results for $\kappa = 0$. (*a*) $\hbar_\theta = -3/4$; (*b*) $\hbar_\theta = -1/2$.

To show the convergence of the series (3.6) when $\kappa = 0$, we investigate the corresponding (5,5), (10,10) and (15,15) Padé expressions (expanded at $Pr = 1$) for $\theta'(0)$ at the 10th, 20th and 30th order of approximations respectively, for $\hbar_\theta = -1/2$. The results are shown in figure 2, together with a comparison to the numerical results. Obviously, as the order of approximation increases, the analytic solution tends to the numerical solution in a larger region of the Prandtl number. Note also that the Padé expression for $\theta'(0)$ at the 20th order of approximation is quite close to that at the 30th-order of approximation. This indicates that the Padé approximation of $\theta'(0)$ at high enough order converges to its 'true' values for the problem considered. Liao (1999*a*) proved in general that a series given by the homotopy analysis method must be a solution of the nonlinear problems considered as long as it is convergent. Thus, if a series given by the homotopy analysis method is convergent, we need not compare it with numerical results.

(*b*) $\kappa = 1/2$

The power series (3.6) for $\kappa = 1/2$ has the same properties as for $\kappa = 0$. For simplicity, we shall not repeat them. In the similar way, we obtain the Padé

Pr	Padé approximation for $\hbar_\theta = -3/4$	Padé approximation for $\hbar_\theta = -1/2$	Numerical results
0.5	0.263	0.236	0.259
1.0	0.333	0.327	0.332
1.5	0.382	0.381	0.382
2.0	0.422	0.422	0.422
2.5	0.456	0.455	0.456
3.0	0.485	0.484	0.485
3.5	0.511	0.510	0.511
4.0	0.535	0.534	0.535
4.5	0.556	0.555	0.557
5.0	0.576	0.576	0.577
7.5	0.661	0.660	0.661
10.0	0.728	0.727	0.728
12.5	0.785	0.784	0.785
15.0	0.834	0.833	0.834
20.0	0.917	0.918	0.918
25.0	0.987	0.989	0.990
30.0	1.046	1.051	1.052
35.0	1.097	1.106	1.107
40.0	1.142	1.155	1.158
45.0	1.183	1.201	1.204
50.0	1.218	1.242	1.247

TABLE 1. Comparisons of the Padé approximation (3.11) for $\theta'(0)$ with numerical results.

expression

$$\theta'(0) = \frac{\sum_{k=0}^{15} b_k (Pr-1)^k}{1 + \sum_{k=1}^{15} b_{15+k}(Pr-1)^k} \tag{3.12}$$

of the corresponding $\theta'(0)$ at the 30th order of approximation when $\hbar_\theta = -2/5$, where the constant coefficients b_k are as follows

$$
\begin{array}{ll}
b_0 = 0.4546022012 & b_{16} = 2.1473844819 \\
b_1 = 1.1520582523 & b_{17} = 2.2996690438 \\
b_2 = 1.3357310819 & b_{18} = 1.6205925656 \\
b_3 = 1.0218081495 & b_{19} = 8.3593303747 \times 10^{-1} \\
b_4 = 5.5921775505 \times 10^{-1} & b_{20} = 3.3261796619 \times 10^{-1} \\
b_5 = 2.3728507964 \times 10^{-1} & b_{21} = 1.0498848057 \times 10^{-1} \\
b_6 = 7.9032245948 \times 10^{-2} & b_{22} = 2.6665644177 \times 10^{-2} \\
b_7 = 2.1345537457 \times 10^{-2} & b_{23} = 5.4774908014 \times 10^{-3} \\
b_8 = 4.6315884555 \times 10^{-3} & b_{24} = 9.0769026331 \times 10^{-4} \\
b_9 = 8.2120012761 \times 10^{-4} & b_{25} = 1.2006191471 \times 10^{-4} \\
b_{10} = 1.1548584434 \times 10^{-4} & b_{26} = 1.2402649562 \times 10^{-5} \\
b_{11} = 1.3034165036 \times 10^{-5} & b_{27} = 9.6080608522 \times 10^{-7} \\
b_{12} = 1.0914320394 \times 10^{-6} & b_{28} = 5.1655937308 \times 10^{-8} \\
b_{13} = 6.8649125917 \times 10^{-8} & b_{29} = 1.6339971000 \times 10^{-9} \\
b_{14} = 2.5034636791 \times 10^{-9} & b_{30} = 1.9007155260 \times 10^{-11} \\
b_{15} = 5.0927717922 \times 10^{-11} &
\end{array}
$$

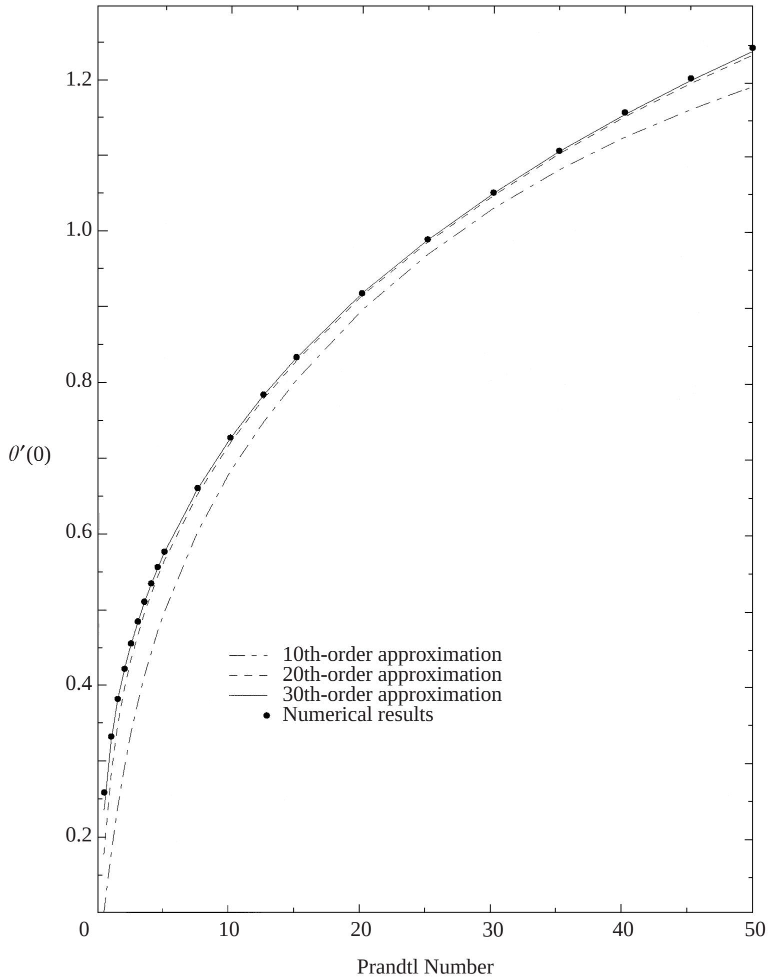

FIGURE 2. Comparisons of numerical results with the Padé expression (3.11) for $\theta'(0)$ for $\kappa = 0$ at different orders of approximation when $\hbar_\theta = -1/2$.

The comparison of (3.12) with the corresponding numerical results of $\theta'(0)$ for $\kappa = 1/2$ is illustrated in figure 3. The good agreement exists in a very large region of the Prandtl number $1 \leqslant Pr \leqslant 50$.

Finally, we point out that there are two ways to obtain the solution of the temperature $\theta(\eta)$. One is to substitute (2.30) into (2.20), which gives the Nth-order approximation

$$\sum_{m=0}^{N} \psi_m = \sum_{k=0}^{N+1} \exp(-k\beta\eta) \sum_{i=0}^{2(N+1-k)} \eta^i \sum_{j=0}^{N} E_{k,i}^{j} (Pr)^j, \tag{3.13}$$

where the coefficient $E_{k,i}^{j}$ is

$$E_{k,i}^{j} = \sum_{l=\max\{j,k-1,i/2+k-1\}}^{N} \Lambda_{l,k}^{i} S_{l,k}^{i,j}. \tag{3.14}$$

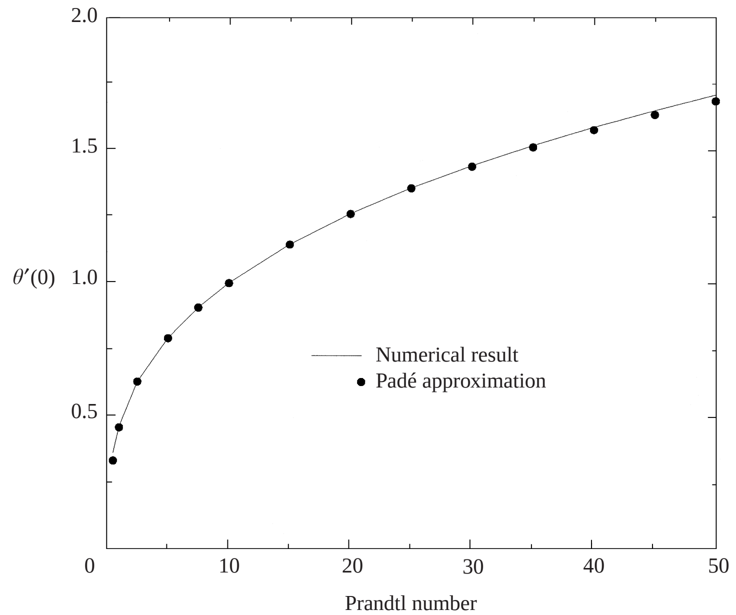

FIGURE 3. Comparisons of numerical results with the Padé expression (3.12) for $\theta'(0)$ for $\kappa = 1/2$ when $\hbar = -2/5$.

Similarly, one can apply the Padé approximation to enlarge the convergence region of each term $\sum_{j=0}^{N} E_{k,i}^{j} (Pr)^j$, too. Another way is to calculate the value of $\theta'(0)$, and then use it and the boundary condition $\theta(0) = 0$ to numerically integrate the original equation (1.5). The latter method greatly simplifies the programming.

4. Discussion and conclusions

We have applied a new analytic technique, namely the homotopy analysis method, to obtain an analytic solution of the temperature distribution $\theta(\eta)$ of laminar viscous flows over a semi-infinite plate. In the general case $T_w(x) - T_\infty = Cx^\kappa$, an explicit solution of the temperature distribution is obtained and the recurrence formulae of the corresponding constant coefficients are given. In view of the importance of $\theta'(0)$, an analytic formula (3.1) for it is obtained. As examples, two special cases are considered. One is the constant temperature distribution ($\kappa = 0$), the other is the constant flux distribution ($\kappa = 1/2$). For each case, a power series about the first-order derivative $\theta'(0)$ of the temperature on the plate at the 30th order of approximation is obtained. The convergence region of these two power series is then greatly enlarged by the Padé approximation. As a result, the analytic formula (3.11) and (3.12) for $\theta'(0)$ agree quite well with numerical results in a very large region of Prandtl number $1 \leqslant Pr \leqslant 50$, as shown in table 1 and figures 1 to 3. To the best of our knowledge, it is the first time that such explicit expressions for $\theta'(0)$ in Prandtl number Pr has been obtained. In the cases of $\kappa = 0$ and $\kappa = 1/2$, the formula (3.11) and (3.12) can be used to give the accurate value of $\theta'(0)$ for any Prandtl number in the region $1 \leqslant Pr \leqslant 50$ without interpolations. Then, knowing the accurate value of $\theta'(0)$, it becomes much easier to obtain the numerical solution of (1.6) with the boundary condition $\theta(0) = 0$. Thus, (3.11) and (3.12) may find wide applications in science and engineering.

Notice that (1.5) and (1.6) have no solution when $Pr = 0$. Thus, Pr cannot be used as a perturbation quantity to obtain perturbation approximations. This is the main reason why there are no explicit perturbation approximations of $\theta'(0)$ in the Prandtl number Pr. This reveals some weaknesses of the perturbation technique: it is too dependent upon the existence of small parameters (perturbation quantities) and in general it is valid only for problems with weak nonlinearity. The homotopy analysis method applied in this paper can overcome all these disadvantages of perturbation techniques. It can be employed whether any small parameters (perturbation quantities) exist or not, and it is valid for strongly nonlinear problems. Using known techniques such as the Padé approximation to greatly enlarge the convergence region of the approximation series given by the homotopy analysis method, one may obtain simple but accurate analytic solutions of many nonlinear problems, as shown in this paper. The success of the homotopy analysis method in solving the set of differential equations of two unknown functions $f(\eta)$ and $\theta(\eta)$ verifies its validity in solving complicated problems. Thus, it appears promising to apply this new method to attack more complicated nonlinear problems in fluid mechanics.

Notice that by using $Pr = 1$ as the centre of expansion we gain a more accurate Padé approximation of $\theta'(0)$, especially for large Pr. This is mainly because (1.5) and (1.6) have no solution when $Pr = 0$ and our result $\theta'(0) = 0$ at $Pr = 0$ has no physical meaning and thus is inaccurate and useless. This shows the importance of considerations from the physical viewpoints.

The homotopy analysis method provides us with great freedom to select the initial guesses, the auxiliary linear operators and values of the auxiliary non-zero parameters $\hbar$ and $\hbar_0$. This kind of freedom is the cornerstone of the validity of the homotopy analysis method. The freedom is so great that we can assume that the solutions are analytic in p and the related series are convergent at $p = 1$ if the initial guesses, the auxiliary linear operators and the auxiliary non-zero parameters are properly selected. Our previous research indicates that there might exist many different initial guesses, different auxiliary linear operators and different values of the auxiliary non-zero parameters, which can ensure the convergence of the series given by the homotopy analysis method. It should be emphasized that Liao (1999*a*) has rigorously proved that all of these different series must be one of the solutions of nonlinear problems under consideration as long as they are convergent. Thus, if a nonlinear problem has a single solution, all of these series converge to this exact/correct solution. In this sense, all of these initial guesses, auxiliary linear operators and values of the auxiliary non-zero parameters are equivalent as long as they can ensure the convergence of the results. Besides, owing to Liao's (1999*a*) mathematical proof, it is even unnecessary to compare our convergent analytic results with numerical ones. So, Liao's (1999*a*) mathematical proof provides a solid base for the validity of the homotopy analysis method. Thus, what we need is to use the freedom mentioned above to select proper initial guesses, proper auxiliary linear operators and so on. Unfortunately, principles underlying how to select such initial guesses and auxiliary linear operators are still unclear and are under investigation. Fortunately, it is also rather difficult to prove that all possible initial guesses and auxiliary linear operators fail to gain a convergent result. So, in practice, the homotopy analysis method can be widely applied as a general technique to many nonlinear problems, exactly like perturbation techniques whose mathematical base is even worse and whose application regions are more narrow. However, from the theoretical viewpoint, a substantial amount of mathematical work on the existence, uniqueness and so on lies ahead, although this is rather difficult.

In summary, the aim of this paper is twofold. First, we give, the first time (to the best of our knowledge), an explicit analytic solution of the temperature distribution of the viscous flow past a semi-infinite flat plate in the general case $T_w(x) - T_\infty = Cx^k$. Also, for the constant wall temperature distribution ($\kappa = 0$) and the constant wall heat flux ($\kappa = 1/2$), an analytic formula for $\theta'(0)$ in the Padé approximation is given, which is valid in a very large region of the Prandtl number, $1 \leqslant Pr \leqslant 50$. All of these analytic formulae may find wide applications in engineering. Second, we show the validity of the homotopy analysis method and its advantages over perturbation techniques for complicated problems having more than one unknown function, which reveals the possibility of employing the homotopy analysis method to attack some famous unsolved nonlinear problems in fluid mechanics, such as the steady-state viscous flow past a sphere (Liao 2002), the plane viscous flow past a circular cylinder and so on. There are many highly nonlinear problems in fluid mechanics. Obviously, a more efficient and easy-to-use analytic technique is beneficial and helpful to deepen our understanding of these complicated nonlinear phenomena, although numerical techniques have been developing rather quickly and becoming more and more important.

The authors would like to thank the reviewers and the editor for their valuable comments and suggestions, especially for mentioning the inaccuracy and non-physical meaning of the result at $Pr = 0$, which put the authors in mind of using $Pr = 1$ as the centre of expansion so as to get the more accurate Padé approximation (3.11) and (3.12).

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